Indian Statistical Institute, Bangalore Centre.
Mid-Semester Exam : Graph Theory

Instructor : Yogeshwaran D.

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Max. points : 50. **Time Limit : 3 hours.**

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly. Answer any five questions. Only the first five answers will be evaluated.

1. (a) If G is a simple graph, show that $diam(G) \geq 3 \Rightarrow diam(\bar{G}) \leq 3$. (**5**)

   (b) Let $2 \leq k \leq n/2$. Consider the graph G with vertices as k-subsets of $[n]$ and edges between pairs of subsets A, B if $|A \cap B| = (k-1)$. Show that $d_G(A,B) = \frac{|A \Delta B|}{2}$ where $A \Delta B$ stands for the symmetric difference of the two sets. (**5**)

2. **Hypercube graph :** The hypercube graph $\mathcal{Q}_n$ is constructed as follows for $n \geq 1$: $V(\mathcal{Q}_n) = \{0,1\}^n$ i.e., each vertex is a string of 0's and 1's of length n. Two vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ share an edge if there exists a co-ordinate $j \in [n]$ such that $u_i = v_i$ for all $i \in [n]/j$ and $u_j \neq v_j$.

   Is this graph a Cayley graph ? If so, describe the group structure. (**5**)

   What is the vertex-Connectivity ($\kappa(\mathcal{Q}_n)$) and edge-connectivity($\lambda(\mathcal{Q}_n)$) of this graph? (**5**)

3. Show that the Petersen graph is the smallest 3-regular graph of girth 5 as well as the largest 3-regular graph of diameter 2. (**10**)

4. Let $d = (d_1, \ldots, d_n)$ be a sequence arranged in non-decreasing order and $d \geq 1$. Show that d is a degree sequence of a forest with exactly k components if and only if $\sum d_i = (2n - 2k)$. If $k = 1$, how many different forests are possible with the degree sequence d ? (**10**)

5. Show that the following algorithm generates a Minimal spanning tree. (**10**)

   Step 1 : Initialize $T = G$.

   Step 2 : Let $e = argmax\{w(e') : e' \in T\}$ (As always break ties arbitrarily).

   Step 3 : If e is not a cut-edge for T (i.e., e is in a cycle in T) then update $T \leftarrow T - e$.

   Step 4 : If T is not a tree, go to Step 2.

   Step 5 : Output $M = T$.

6. Let M, M' be minimal spanning trees of the graph G with edge-weights $w(.)$. Show that for any $s \geq 0$,

$$|\{e \in M : w(e) = s\}| = |\{e \in M' : w(e) = s\}|. \ (\mathbf{10})$$

7. Consider the network flow problem with the following edge capacities, $c(u, v)$ for edge (u, v): $c(s, 2) = 2, c(s, 3) = 13, c(2, 5) = 12, c(3, 4) = 5, c(3, 7) = 6, c(4, 5) = 1, c(4, 6) = 1, c(6, 5) = 2, c(6, 7) = 3, c(5, t) = 6, c(7, t) = 2$. As usual, s denotes the source and t denotes the sink.

   (a) Draw the network with the flows on each edge marked. (**2**)
   (b) Run the Ford-Fulkerson algorithm to find the maximum flow. Draw/mention the flows at each step of the algorithm. (**5**)
   (c) What is the minimal cut ? (**3**)